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A family of tridiagonal pairs and related symmetric functions

Pascal Baseilhac

Laboratoire de Mathématiques et Physique Théorique CNRS/UMR 6083, Fédération Denis Poisson, Université de Tours, Parc de Grandmont, 37200 Tours, France

E-mail: baseilha@phys.univ-tours.fr

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Abstract
A family of tridiagonal pairs which appear in the context of quantum integrable systems is studied in detail. The corresponding eigenvalue sequences, eigenspaces and the block tridiagonal structure of their matrix realizations with respect the dual eigenbasis are described. The overlap functions between the two dual bases are shown to satisfy a coupled system of recurrence relations and a set of discrete second-order q-difference equations which generalize those associated with the Askey–Wilson orthogonal polynomials with a discrete argument. Normalizing the fundamental solution to unity, the hierarchies of solutions are rational functions of one discrete argument, explicitly derived in some simplest examples. The weight function which ensures the orthogonality of the system of rational functions defined on a discrete real support is given.

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1. Introduction

Jacobi matrices deserve continuous attention in operator theory, as they play an important role in various areas such as numerical analysis, orthogonal polynomials, continued fractions. They also find applications in mathematical physics: they are discrete analogues of second-order linear differential operators of Schrödinger type on the half-line, and appear in integrable systems and the theory of random matrices.

An important domain of application concerns the theory of orthogonal polynomials of one argument x—denoted by $p_n(x)$ below. Indeed, it is well known that every sequence $\{p_n\}_{n=0}^{n=\infty}$ satisfies a 3-term relation

$$x p_n = b_n p_{n+1} + a_n p_n + c_n p_{n-1}, \qquad n = 1, 2, \ldots \tag{1}$$

where $p_0 \equiv 1$, $c_0 \equiv 0$ by definition, and $a_n, b_n, c_n \in \mathbb{C}$. A Jacobi matrix J being a tridiagonal matrix of either finite or infinite dimension, in the infinite-dimensional case the non-vanishing coefficients a_n, b_n, c_n yield to consider the spectral problem associated with the sequence $\{p_n\}_{n=0}^{n=\infty}$:

$$
Jp_n(x) = xp_n(x) \qquad \text{with} \qquad J = \begin{pmatrix} a_0 & c_1 & & & 0 \\ b_0 & a_1 & c_2 & & \\ & b_1 & a_2 & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \\ 0 & & & & \end{pmatrix}, \tag{2}
$$

where $p_0(x), p_1(x), p_2(x), \ldots$ form a basis of $\mathbb{C}[x]$. In the finite-dimensional case, the main submatrix of J which is called the truncated matrix and denoted by $J^{(N)}$ is usually introduced. If it is of size $N+1 \times N+1$, then the zeros of $p_{N+1}(x)$ coincide with the $N+1$ distinct eigenvalues of $J^{(N)}$. In other words, $p_{N+1}(x)$ is proportional to the characteristic polynomial of $J^{(N)}$ and x takes discrete values. As a consequence, the corresponding system of orthogonal polynomials is defined on a discrete support.

Among the families of orthogonal polynomials in one variable, only those satisfying (2) and a second-order differential equation are known to be the Jacobi, Hermite, Laguerre and Bessel polynomials, as shown in a theorem of Bochner [1]. This led the authors of [2] to derive a 'q-version' of this theorem using an operator identity of independent interest. For this family of orthogonal polynomials satisfying (2) and a second-order q-difference equation of the form

$$
a(y)(p_n(s(qy)) - p_n(s(y))) + b(y)(p_n(s(q^{-1}y)) - p_n(s(y))) = \theta_n p_n(s(y)) \quad \text{for} \quad n \geqslant 0, \tag{3}
$$

where the argument is either $x \equiv s(y) = (y + y^{-1})/2$ or $x \equiv s(y) = y$, the only solutions to (2) and (3) are given by the Askey–Wilson polynomials [3] or the big q-Jacobi polynomials, respectively [2].

In view of the tridiagonal structure of the lhs of above equations, it is natural to study the algebraic structure associated with a pair of Jabobi matrices having a spectral problem of the form (2), (3). In particular, there have been different ways to characterize the Askey–Wilson polynomials and related q-hypergeometric functions from this *algebraic* point of view. For instance, let us mention [4] where finite-dimensional representations of the quantum group $U_q(sl_2)$ arise, and more generally [5] where finite-dimensional representations of the Askey–Wilson algebra $AW(3)$ are introduced and studied in detail.

In the last few years, a unified algebraic framework based on the concept of *Leonard pair* has been introduced in connection with the orthogonal polynomials of the Askey scheme. Roughly speaking (details can be found in [6]), a Leonard pair A, A^* acting on a finite-dimensional representation V is such that A, A^* are diagonalizable where the matrix representing A^* (resp. A) is irreducible tridiagonal in the basis which diagonalizes A (resp. A^*). In particular, given a Leonard pair there exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ taken from an arbitrary field $\mathbb{K}$ such that [7]

$$
\begin{aligned}
&A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* = \gamma^*A^2 + \omega A + \eta I, \\
&A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^*A = \gamma A^{*2} + \omega A^* + \eta^* I.
\end{aligned} \tag{4}
$$

The sequence is uniquely determined by the Leonard pair provided the dimension of V is at least 4, and the equations above are called the *Askey–Wilson relations* (see also [5]). Remarkably, all known examples of Leonard pairs for $\beta = q + q^{-1}$ and $q \neq -1$ are related with orthogonal polynomials of the Askey scheme [8] in the following sense: the entries of the transition matrix (sometimes called the overlap coefficients) relating the two 'dual' bases which diagonalize A, A^*, respectively, can be expressed in terms of one of the following orthogonal polynomials (q-hypergeometric functions): Racah (${}_4F_3$), Hahn and dual Hahn (${}_3F_2$), Krawtchouk (${}_2F_1$), q-Racah (${}_4\phi_3$), q-Hahn and dual q-Hahn (${}_3\phi_2$), q-Krawtchouk—classical, affine, quantum, dual—(${}_2\phi_1$).

Given the connection between Leonard pairs and the polynomials of the Askey-scheme, several features led to consider a more general object called a *Tridiagonal* (TD) pair [9]. From an algebraic point of view, tridiagonal pairs play an important role in the representation theory of the tridiagonal algebra [9–11]. This associative algebra $\mathbb{T}$ with unity consists of two generators—called standard generators—acting on a vector space V, say $\mathsf{A} : V \to V$ and $\mathsf{A}^* : V \to V$. In general, the defining relations of $\mathbb{T}$ depend on five scalars $\rho, \rho^*, \gamma, \gamma^*$ and β. In the following, we will focus on the *reduced* parameter sequence $\gamma = 0, \gamma^* = 0$ and $\beta = q + q^{-1}$ which exhibits all interesting properties that can be extended to more general parameter sequences. In this case, the so-called *tridiagonal relations* take the form

$$[\mathsf{A}, [\mathsf{A}, [\mathsf{A}, \mathsf{A}^*]_q]_{q^{-1}}] = \rho[\mathsf{A}, \mathsf{A}^*], \qquad [\mathsf{A}^*, [\mathsf{A}^*, [\mathsf{A}^*, \mathsf{A}]_q]_{q^{-1}}] = \rho^*[\mathsf{A}^*, \mathsf{A}], \tag{5}$$

where the q-commutator $[\mathsf{A}, \mathsf{B}]_q = q^{1/2}\mathsf{AB} - q^{-1/2}\mathsf{BA}$ has been introduced. Here, q is a deformation parameter assumed to be not a root of unity. Note that for $\rho = \rho^* = 0$ the relations (5) reduce to q-Serre relations, or for $q = 1, \rho = \rho^* = 16$ they coincide with the Dolan–Grady relations [12].

The purpose of this paper is to investigate a family of TD pairs recently discovered in the context of quantum integrable systems and related algebraic structures[1]. There are some reasons to do so [10], appart from the applications to integrable systems that will be considered separately [16]. Here, we study in detail their matrix representations: eigenvalue sequences, eigenvectors and block tridiagonal stucture with respect to the dual basis. Using these data, we identify a family of coupled recurrence relations and a set of q-difference equations generalizing (1), (3), and describe their explicit solutions in the simplest cases. Our results are in agreement with known ones [9, 10] and provide further understanding of TD pairs. In particular, they suggest that the tridiagonal algebra (5) may provide a classification scheme for orthogonal symmetric functions of one argument.

Convention: In this paper, $\mathbb{R}, \mathbb{C}$ denote the field of real and complex numbers, respectively, $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$, $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. For convenience, the deformation parameter is sometimes denoted by $q \equiv \exp(\phi)$ with $\phi \in \mathbb{C}$. According to a common notational convention, for a linear transformation A its conjugate transpose is denoted by A^*. Note that we do not use this convention. Here, the conjugate transpose of A is denoted by $\mathsf{A}^\dagger$.

2. A family of tridiagonal pairs

Suppose that each linear transformation A, A^* is diagonalizable on V i.e. V is spanned by the eigenspaces of A or A^*. A general object called a *Tridiagonal pair* is defined as follows:

Definition 2.1. *[9] Let V denote a vector space over an arbitrary field $\mathbb{K}$ with finite positive dimension. By a Tridiagonal pair (or TD pair) on V we mean an ordered pair A, A^*, where*

[1] The tridiagonal algebra with relations (5) is the quantum group structure behind the reflection equation associated with $U_{q^{1/2}}(\widehat{sl_2})$ [13, 14]. It also plays an important role as a fundamental non-Abelian symmetry in some integrable systems: it generates a hierarchy of mutually commuting quantities which ensure the integrability of the system. Among the examples of integrable systems which enjoy this symmetry, one finds for instance the boundary sine-Gordon model or the XXZ open spin chain with general boundary conditions [13–15].

$\mathsf{A} : V \to V$ and $\mathsf{A}^* : V \to V$ are linear transformations satisfying (i)–(iv) below.

(1) A and A^* are both diagonalizable on V.

(2) There exists an ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of A such that

$$\mathsf{A}^* V_n \subseteq V_{n-1} + V_n + V_{n+1} \qquad (0 \leqslant n \leqslant d), \tag{6}$$

where $V_{-1} = 0$, $V_{d+1} = 0$.

(3) There exists an ordering $V_0^*, V_1^*, \ldots, V_\delta^*$ of the eigenspaces of A^* such that

$$\mathsf{A} V_s^* \subseteq V_{s-1}^* + V_s^* + V_{s+1}^* \qquad (0 \leqslant s \leqslant \delta), \tag{7}$$

where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.

(4) There is no subspace W of V such that both $\mathsf{A}W \subseteq W$, $\mathsf{A}^*W \subseteq W$, other than $W = 0$ and $W = V$.

We say the above TD pair is over $\mathbb{K}$.

Although a complete classification of tridiagonal pairs is not known yet, the following results have been obtained. Referring to the TD pair in the definition above, as shown in [9, lemma 4.2] one has $d = \delta$ which is called the *diameter*. Also, for $0 \leqslant n \leqslant d$, the dimensions of the eigenspaces V_n and V_s^* are equal and the sequence $(\dim(V_0), \dim(V_1), \ldots, \dim(V_d))$—called the *shape vector*—is symmetric and unimodal i.e. [17]

$$\dim(V_n) = \dim(V_{d-n}) \quad \text{for} \quad 0 \leqslant n \leqslant d, \qquad \dim(V_{n-1}) \leqslant \dim(V_n) \quad \text{for} \quad 1 \leqslant n \leqslant d/2. \tag{8}$$

Among the known examples of TD pairs, one finds a subset such that A, A^* have eigenspaces of dimension one i.e. with a shape vector $(1, 1, \ldots, 1)$. These are called Leonard pairs, classified in [6]. In particular, they satisfy (for details, see [7]) the Askey–Wilson (AW) relations (4) first introduced by Zhedanov [5]. Other examples of TD pairs are for instance the subset corresponding to $\rho = \rho^* = 0$ which reduce (5) to the q-Serre relations of $U_{q^{1/2}}(\widehat{sl_2})$. In this case, the shape vector associated with the TD pair A, A^* is such that

$$\dim(V_n) \leqslant \binom{d}{n}, \qquad 0 \leqslant n \leqslant d \quad \text{where} \quad \binom{d}{n} = \frac{d!}{n!(d-n)!} \tag{9}$$

denotes the binomial coefficient. It is important to mention that TD pairs also arise from irreducible finite-dimensional representations of the Lie algebra sl_2 and the Onsager algebra [18].

From now on, we focus our attention on the $U_{q^{1/2}}(\widehat{sl_2})$ algebra which plays a crucial role in the following analysis and we set $\mathbb{K} = \mathbb{C}$. The quantum Kac–Moody algebra $U_{q^{1/2}}(\widehat{sl_2})$ is generated by the elements $\{H_j, E_j, F_j\}$, $j \in \{0, 1\}$. Denoting the entries of the extended Cartan matrix[2] by $\{a_{ij}\}$, they satisfy the commutation relations

$$[H_i, H_j] = 0, \qquad [H_i, E_j] = a_{ij} E_j,$$
$$[H_i, F_j] = -a_{ij} F_j, \qquad [E_i, F_j] = \delta_{ij} \frac{q^{H_i/2} - q^{-H_i/2}}{q^{1/2} - q^{-1/2}}$$

together with the q-Serre relations

$$[E_i, [E_i, [E_i, E_j]_q]_{q^{-1}}] = 0, \qquad \text{and} \qquad [F_i, [F_i, [F_i, F_j]_q]_{q^{-1}}] = 0. \tag{10}$$

[2] With $i, j \in \{0, 1\}$: $a_{ii} = 2$, $a_{ij} = -2$ for $i \neq j$.

The sum $K = H_0 + H_1$ is the central element of the algebra. The Hopf algebra structure is ensured by the existence of a co-multiplication $\Delta : U_{q^{1/2}}(\widehat{sl_2}) \to U_{q^{1/2}}(\widehat{sl_2}) \otimes U_{q^{1/2}}(\widehat{sl_2})$ and a counit $\mathcal{E} : U_{q^{1/2}}(\widehat{sl_2}) \to \mathbb{C}$ with

$$\begin{aligned}
\Delta(E_i) &= E_i \otimes q^{H_i/4} + q^{-H_i/4} \otimes E_i, \\
\Delta(F_i) &= F_i \otimes q^{H_i/4} + q^{-H_i/4} \otimes F_i, \\
\Delta(H_i) &= H_i \otimes \mathbb{I} + \mathbb{I} \otimes H_i
\end{aligned} \tag{11}$$

and

$$\mathcal{E}(E_i) = \mathcal{E}(F_i) = \mathcal{E}(H_i) = 0, \qquad \mathcal{E}(\mathbb{I}) = 1.$$

More generally, one defines the N-coproduct $\Delta^{(N)} : U_{q^{1/2}}(\widehat{sl_2}) \longrightarrow U_{q^{1/2}}(\widehat{sl_2}) \otimes \cdots \otimes U_{q^{1/2}}(\widehat{sl_2})$ as

$$\Delta^{(N)} \equiv (\mathrm{id} \times \cdots \times \mathrm{id} \times \Delta) \circ \Delta^{(N-1)} \tag{12}$$

for $N \geqslant 3$ with $\Delta^{(2)} \equiv \Delta$, $\Delta^{(1)} \equiv \mathrm{id}$. The opposite N-coproduct $\Delta'^{(N)}$ is similarly defined with $\Delta' \equiv \sigma \circ \Delta$ where the permutation map $\sigma(x \otimes y) = y \otimes x$ for all $x, y \in U_{q^{1/2}}(\widehat{sl_2})$ is used.

In order to construct explicitly a family of TD pairs, we use the link recently exhibited [13, 14] between the tridiagonal algebra $\mathbb{T}$ and a class of quadratic algebra, namely the reflection equation introduced in [19, 20]. Based on [13, 14], we restrict our attention to the quantum Kac–Moody algebra $U_{q^{1/2}}(\widehat{sl_2})$ with zero centre i.e. $K \equiv 0$. In this special case, this algebra with defining relations (10) is called the quantum loop algebra of sl_2 denoted by $U_{q^{1/2}}(\mathcal{L}(sl_2))$ below. We have the following result (see also [13, 14]):

Proposition 2.2. *Let $\{k_+, k_-, \epsilon_\pm\}$ denote nonzero scalars in $\mathbb{C}$. There is an algebra homomorphism $\mathbb{T} \mapsto U_{q^{1/2}}(\mathcal{L}(sl_2))$ such that*

$$\begin{aligned}
\textsf{A} &\mapsto k_+ E_1 q^{H_1/4} + k_- F_1 q^{H_1/4} + \epsilon_+ q^{H_1/2} \\
\textsf{A}^* &\mapsto k_- E_0 q^{H_0/4} + k_+ F_0 q^{H_0/4} + \epsilon_- q^{H_0/2}
\end{aligned} \tag{13}$$

with

$$\rho = \rho^* = (q^{1/2} + q^{-1/2})^2 k_+ k_-. \tag{14}$$

Proof. Replacing (13) in (5), we use the defining relations and the q-Serre relations (10) to simplify the lhs of the tridiagonal relations. Straightforward calculations show that the lhs of (5) reduces to the r.h.s provided $\rho = \rho^*$ is fixed in terms of the deformation parameter q, with the relation given above. □

Remark 2.3. This realization is different from that proposed by Ito and Terwilliger in [17] in which case $\rho = 0$, i.e. (5) reduce to q-Serre relations.

Remark 2.4. Using the homorphism defined above, it follows [14] that the corresponding tridiagonal relations (5) are invariant under the action of the coproduct (11) as well as its generalization (12).

To obtain finite-dimensional irreducible representations of the standard generators (13), one introduces the evaluation homomorphism $\pi_v : U_{q^{1/2}}(\mathcal{L}(sl_2)) \mapsto U_{q^{1/2}}(sl_2))$ in the principal gradation:

$$\begin{aligned}
\pi_v[E_1] &= vS_+, & \pi_v[E_0] &= vS_-, \\
\pi_v[F_1] &= v^{-1}S_-, & \pi_v[F_0] &= v^{-1}S_+, \\
\pi_v[q^{H_1/2}] &= q^{s_3}, & \pi_v[q^{H_0/2}] &= q^{-s_3},
\end{aligned}$$

where the generators of $U_{q^{1/2}}(sl_2)$ satisfy

$$[s_3, S_\pm] = \pm S_\pm \qquad \text{and} \qquad [S_+, S_-] = (q^{s_3} - q^{-s_3})/(q^{1/2} - q^{-1/2}).$$

As shown in [21], every finite-dimensional irreducible representation of $U_{q^{1/2}}(\mathcal{L}(sl_2))$ is a tensor product of evaluation representations. Using the N-coproduct (12), it follows [13, 14]:

Lemma 2.5. *Let $k_+ = (k_-)^\dagger, \epsilon_\pm$ nonzero scalars in $\mathbb{C}$. For any $v_i \in \mathbb{C}^*$ with integers $i = 1, \ldots, N$, let the pair of elements $\mathsf{W}_0^{(N)}, \mathsf{W}_1^{(N)}$ defined by*

$$\begin{aligned} \mathsf{W}_0^{(N)} &= \big(k_+ v_N q^{1/4} S_+ q^{s_3/2} + k_- v_N^{-1} q^{-1/4} S_- q^{s_3/2}\big) \otimes \mathbb{I}^{(N)} + q^{s_3} \otimes \mathsf{W}_0^{(N-1)}, \\ \mathsf{W}_1^{(N)} &= \big(k_+ v_N^{-1} q^{-1/4} S_+ q^{-s_3/2} + k_- v_N q^{1/4} S_- q^{-s_3/2}\big) \otimes \mathbb{I}^{(N)} + q^{-s_3} \otimes \mathsf{W}_1^{(N-1)} \end{aligned} \tag{15}$$

with $\mathsf{W}_0^{(0)} \equiv \epsilon_+$, $\mathsf{W}_1^{(0)} \equiv \epsilon_-$. Let V denote a finite-dimensional N-tensor product representation of $\otimes_1^N U_{q^{1/2}}(\mathcal{L}(sl_2))$. Assume V is an irreducible $\big(\mathsf{W}_0^{(N)}, \mathsf{W}_1^{(N)}\big)$-module and each of $\mathsf{W}_0^{(N)}, \mathsf{W}_1^{(N)}$ is diagonalizable on V. Then the pair $\mathsf{W}_0^{(N)}, \mathsf{W}_1^{(N)}$ acts on V as a tridiagonal pair.

Proof. Conditions 1 and 4 of definition 2.1 hold by assumption. From proposition 2.2, in the evaluation representation of $U_{q^{1/2}}(\mathcal{L}(sl_2))$ the elements $\mathsf{W}_0^{(1)}, \mathsf{W}_1^{(1)}$ satisfy the tridiagonal relations (13). Using the N-coproduct (12) and the result of [21], it follows that $\mathsf{W}_0^{(N)}, \mathsf{W}_1^{(N)}$ satisfy the tridiagonal relations (13) with (14) for general values of N. Then, according to [10, theorem 10] the assertion follows. □

3. Matrix representation and a block tridiagonal structure

For general values of q, irreducible $(2j+1)$-finite-dimensional representations of $U_{q^{1/2}}(sl_2)$ are well known. In the following, for simplicity we will focus on the case $j = 1/2$ and introduce the canonical basis[3] $\{f_+^i, f_-^i\}$ for the ith two-dimensional representation in the N-tensor product representation of $\otimes_1^N U_{q^{1/2}}(sl_2)$. Defining the endomorphism $\xi : U_{q^{1/2}}(sl_2) \mapsto \mathrm{End}(\mathbb{C}^2)$, in this basis one has in terms of the Pauli matrices $\sigma_\pm, \sigma_3$:

$$S_+ \mapsto \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad S_- \mapsto \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad s_3 \mapsto \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{16}$$

Definition 3.1. *Define $k_+ = (k_-)^\dagger = -(q^{1/2} - q^{-1/2})\,\mathrm{e}^{\mathrm{i}\theta}/2$ with $\theta \in \mathbb{R}$, $q = \mathrm{e}^{\phi}$ with ϕ purely imaginary, and $\{\alpha, \alpha^*\} \in \mathbb{C}^*$. Let $V = (\mathbb{C}^2)^{\otimes N}$. Define $\mathcal{W}_0^{(0)} \equiv \epsilon_+ = \cosh\alpha$, $\mathcal{W}_1^{(0)} \equiv \epsilon_- = \cosh\alpha^*$. We define the family of matrices of size $2^N \times 2^N$:*

$$\begin{aligned} \mathcal{W}_0^{(N)} &= (k_+\sigma_+ + k_-\sigma_-) \otimes \mathbb{I}^{(N-1)} + q^{\sigma_3/2} \otimes \mathcal{W}_0^{(N-1)}, \\ \mathcal{W}_1^{(N)} &= (k_+\sigma_+ + k_-\sigma_-) \otimes \mathbb{I}^{(N-1)} + q^{-\sigma_3/2} \otimes \mathcal{W}_1^{(N-1)}. \end{aligned} \tag{17}$$

Below, we study in detail the structure of these matrices for general values of N in their eigenbasis and dual. Although it corresponds to the spin $-\, j = 1/2$ case, the structure of the pair $\mathcal{W}_0^{(N)}, \mathcal{W}_1^{(N)}$ can be studied along the same line for higher representations leading to analogous results.

[3] The scalar product $\langle .\,, . \rangle$ is defined by $\langle f_\pm^j, f_\pm^j \rangle \equiv 1$, $\langle f_\pm^j, f_\mp^j \rangle \equiv 0$ for any integer j.

3.1. Example $N = 1$ and the Askey–Wilson algebra

As pointed out in [13], for $N = 1$ the elements $\mathsf{W}_0^{(1)}, \mathsf{W}_1^{(1)}$ satisfy the Askey–Wilson algebra (4) introduced in [5]. In particular, their matrix representations of size $2j + 1 \times 2j + 1$ provide an example of Leonard pairs [7] with shape vector $(1, 1, \ldots, 1)$. For the simplest case $j = 1/2$, $\mathcal{W}_0^{(1)}, \mathcal{W}_1^{(1)}$ are 2×2 matrices, whose entries in the canonical basis follow from (17) with (16). Let $\psi_n^{(1)}, n = 0, 1$ denote the two eigenvectors of $\mathcal{W}_0^{(1)}$ in V with (multiplicity free) eigenvalues $\lambda_n^{(1)}$. One finds $\lambda_n^{(1)} = \cosh(\alpha + (1 - 2n)\phi/2), n = 0, 1$ with

$$\psi_0^{(1)} = \mathrm{e}^{\alpha+\mathrm{i}\theta} f_+^1 + f_-^1, \qquad \psi_1^{(1)} = \mathrm{e}^{-\alpha+\mathrm{i}\theta} f_+^1 + f_-^1. \tag{18}$$

Assuming α generic, they are independent of each other. In this new basis, the element $\mathcal{W}_1^{(1)}$ can be written as

$$\mathcal{W}_1^{(1)} = \begin{pmatrix} a_0^{(1)} & c_1^{(1)} \\ b_0^{(1)} & a_1^{(1)} \end{pmatrix}, \tag{19}$$

where

$$a_0^{(1)} = \frac{\cosh\alpha^* \sinh(\alpha - \phi/2) - \sinh(\phi/2)}{\sinh\alpha}, \qquad a_1^{(1)} = \frac{\cosh\alpha^* \sinh(\alpha + \phi/2) + \sinh(\phi/2)}{\sinh\alpha},$$
$$b_0^{(1)} = \frac{\mathrm{e}^{\alpha}(\cosh\alpha + \cosh\alpha^*)\sinh(\phi/2)}{\sinh\alpha}, \qquad c_1^{(1)} = -\frac{\mathrm{e}^{-\alpha}(\cosh\alpha + \cosh\alpha^*)\sinh(\phi/2)}{\sinh\alpha}.$$

By analogy, for generic $\alpha^* \in \mathbb{C}$ the eigenvectors of the element $\mathcal{W}_1^{(1)}$—denoted by $\varphi_s^{(1)}$ with $s = 0, 1$—form a complete basis of V. They are easily obtained substituting

$$n \to s, \quad \alpha \leftrightarrow -\alpha^*, \quad \phi \to -\phi, \quad \theta \to \theta + \pi \tag{20}$$

in (18). In this dual basis, the matrix $\mathcal{W}_1^{(1)}$ is diagonal with eigenvalues $\lambda_s^{(1)} = \cosh(\alpha^* + (1-2s)\phi/2)$, $s = 0, 1$ whereas the entries $\{\tilde{a}_s^{(1)}, \tilde{b}_s^{(1)}, \tilde{c}_s^{(1)}\}$ of $\mathcal{W}_0^{(1)}$ follow from $\{a_n^{(1)}, b_n^{(1)}, c_n^{(1)}\}$ using (20).

3.2. Example $N = 2$

Contrary to the case $N = 1$, the elements $\mathsf{W}_0^{(2)}, \mathsf{W}_1^{(2)}$ do not satisfy the defining relations (4) of the Askey–Wilson algebra [14]. Instead, they satisfy a pair of fifth order relations that can be found in [15]. Also, the eigenvalues of the (4×4)-dimensional matrices $\mathcal{W}_0^{(2)}, \mathcal{W}_1^{(2)}$ are not mutually distinct. More precisely, the shape vector associated with these eigenvalues $\lambda_n^{(2)} = \cosh(\alpha + (2 - 2n)\phi/2)$ for $n = 0, 1, 2$ takes the form $(\dim(V_0), \dim(V_1), \dim(V_2)) = (1, 2, 1)$. For generic values of the parameters α, ϕ, θ the corresponding eigenvectors denoted by $\psi_{n[i]}^{(2)}$ form a complete basis. In the canonical basis $\left(f_\pm^2 \otimes f_\pm^1\right)$ of $U_{q^{1/2}}(sl_2) \otimes U_{q^{1/2}}(sl_2)$, they take the form

$$\begin{aligned}
\psi_{0[1]}^{(2)} &= \left(\exp(\alpha + \phi/2 + \mathrm{i}\theta) f_+^2 + f_-^2\right) \otimes \left(\exp(\alpha + \mathrm{i}\theta) f_+^1 + f_-^1\right), \\
\psi_{1[1]}^{(2)} &= \left(\exp(\alpha - \phi/2 + \mathrm{i}\theta) f_+^2 + f_-^2\right) \otimes \left(\exp(-\alpha + \mathrm{i}\theta) f_+^1 + f_-^1\right), \\
\psi_{1[2]}^{(2)} &= \left(\exp(-\alpha - \phi/2 + \mathrm{i}\theta) f_+^2 + f_-^2\right) \otimes \left(\exp(\alpha + \mathrm{i}\theta) f_+^1 + f_-^1\right), \\
\psi_{2[1]}^{(2)} &= \left(\exp(-\alpha + \phi/2 + \mathrm{i}\theta) f_+^2 + f_-^2\right) \otimes \left(\exp(-\alpha + \mathrm{i}\theta) f_+^1 + f_-^1\right),
\end{aligned} \tag{21}$$

where the index $i = 1, 2$ which characterizes the multiplicity of the eigenvalues has been introduced. With respect to the basis $\psi^{(2)}_{0[1]}, \psi^{(2)}_{1[1]}, \psi^{(2)}_{1[2]}, \psi^{(2)}_{2[1]}$ the matrices representing $\mathcal{W}^{(2)}_0, \mathcal{W}^{(2)}_1$ read, respectively,

$$\mathcal{W}^{(2)}_0 = \mathrm{diag}\big(\lambda^{(2)}_0, \lambda^{(2)}_1, \lambda^{(2)}_1, \lambda^{(2)}_2\big), \qquad \mathcal{W}^{(2)}_1 = \begin{pmatrix} a^{(2)}_{0[11]} & c^{(2)}_{1[11]} & c^{(2)}_{1[12]} & 0 \\ b^{(2)}_{0[11]} & a^{(2)}_{1[11]} & a^{(2)}_{1[12]} & c^{(2)}_{2[11]} \\ b^{(2)}_{0[21]} & a^{(2)}_{1[21]} & a^{(2)}_{1[22]} & c^{(2)}_{2[21]} \\ 0 & b^{(2)}_{1[11]} & b^{(2)}_{1[12]} & a^{(2)}_{2[11]} \end{pmatrix}, \tag{22}$$

where the entries are

$$a^{(2)}_{0[11]} = \frac{a^{(1)}_0 \sinh\alpha - \sinh(\phi/2)}{\sinh(\alpha+\phi/2)},$$

$$a^{(2)}_{1[11]} = \frac{a^{(1)}_1 \sinh(\alpha-\phi) - \sinh(\phi/2)}{\sinh(\alpha-\phi/2)}, \qquad a^{(2)}_{1[12]} = -\frac{\mathrm{e}^{-\alpha}\sinh\phi}{\sinh(\alpha-\phi/2)} b^{(1)}_0,$$

$$a^{(2)}_{1[21]} = \frac{\mathrm{e}^{\alpha}\sinh\phi}{\sinh(\alpha+\phi/2)} c^{(1)}_1, \qquad a^{(2)}_{1[22]} = \frac{a^{(1)}_0 \sinh(\alpha+\phi) + \sinh(\phi/2)}{\sinh(\alpha+\phi/2)},$$

$$a^{(2)}_{2[11]} = \frac{a^{(1)}_1 \sinh\alpha + \sinh(\phi/2)}{\sinh(\alpha-\phi/2)},$$

$$b^{(2)}_{0[11]} = \mathrm{e}^{\phi/2} b^{(1)}_0, \qquad b^{(2)}_{0[21]} = \frac{\mathrm{e}^{\alpha+\phi/2}\sinh(\phi/2)}{\sinh(\alpha+\phi/2)}\big(a^{(1)}_0 + \cosh(\alpha+\phi/2)\big),$$

$$b^{(2)}_{1[11]} = \frac{\mathrm{e}^{\alpha-\phi/2}\sinh(\phi/2)}{\sinh(\alpha-\phi/2)}\big(a^{(1)}_1 + \cosh(\alpha-\phi/2)\big), \qquad b^{(2)}_{1[12]} = \mathrm{e}^{-\phi/2}\frac{\sinh(\alpha+\phi/2)}{\sinh(\alpha-\phi/2)} b^{(1)}_0,$$

$$c^{(2)}_{2[11]} = -\frac{\mathrm{e}^{-\alpha+\phi/2}\sinh(\phi/2)}{\sinh(\alpha-\phi/2)}\big(a^{(1)}_1 + \cosh(\alpha-\phi/2)\big), \qquad c^{(2)}_{2[21]} = \mathrm{e}^{\phi/2} c^{(1)}_1,$$

$$c^{(2)}_{1[11]} = \mathrm{e}^{-\phi/2}\frac{\sinh(\alpha-\phi/2)}{\sinh(\alpha+\phi/2)} c^{(1)}_1, \qquad c^{(2)}_{1[12]} = -\frac{\mathrm{e}^{-\alpha-\phi/2}\sinh(\phi/2)}{\sinh(\alpha+\phi/2)}\big(a^{(1)}_0 + \cosh(\alpha+\phi/2)\big).$$

The set of eigenvectors $\varphi^{(2)}_{s[k]}$ that diagonalize $\mathcal{W}^{(1)}_1$ with $s = 0, 2$ $(k = 1)$ and $s = 1$ $(k = 1, 2)$ are obtained using (20) in (21), and their eigenvalues are $\lambda^{(2)}_s = \cosh(\alpha^* + (2-2s)\phi/2), s = 0, 1, 2$. In this dual basis, the matrix $\mathcal{W}^{(2)}_0$ takes a block tridiagonal form similar to $\mathcal{W}^{(2)}_1$ in (22), where the corresponding entries $\big\{\tilde{a}^{(2)}_s, \tilde{b}^{(2)}_s, \tilde{c}^{(2)}_s\big\}$ follow from $\big\{a^{(2)}_n, b^{(2)}_n, c^{(2)}_n\big\}$ using (20). Assuming that there are no special relations between the parameters α, ϕ, θ or α^*, ϕ, θ, it follows from above results that the matrices $\mathcal{W}^{(2)}_0, \mathcal{W}^{(2)}_1$ satisfy the requirements of definition 2.1. The block tridiagonal structure, which did not appear for $N = 1$ is now clear from (22). In the next section, we show that this property generalizes for higher values of N.

3.3. Generalization

To describe the family of matrices $\mathcal{W}^{(N)}_0, \mathcal{W}^{(N)}_1$ in detail, first we need to construct explicitly a complete basis of eigenvectors $\psi^{(N)}_{n[i]}$ that diagonalize $\mathcal{W}^{(N)}_0$ for any N, and identify the corresponding eigenvalues.

Definition 3.2. *Let V denote the finite-dimensional representation (of dimension 2^N) on which $\mathcal{W}^{(N)}_0$ acts. Suppose $\mathcal{W}^{(N)}_0$ is diagonalizable on V, and let $V_0, V_1, V_2, \ldots$ denote a standard* *ordering of the eigenspaces of* $\mathcal{W}_0^{(N)}$. *For* $n = 0, 1, 2, \ldots,$ *let* $\lambda_n^{(N)}$ *denote the eigenvalue sequence of* $\mathcal{W}_0^{(N)}$.

The structure of the eigenvectors and eigenvalues is as follows:

Proposition 3.3. *Define* $k_+ = (k_-)^\dagger = -(q^{1/2} - q^{-1/2})\,\mathrm{e}^{\mathrm{i}\theta}/2$. *Let* $\{\alpha, \alpha^*\} \in \mathbb{C}^*, \theta \in \mathbb{R}$ *be generic scalars and* ϕ *purely imaginary. Introduce the set* $\epsilon_j^{[i]} = \pm 1$, $j = 1, \ldots, N$ *and* $i \in \{1, \ldots, \dim(V_n)\}$. *The matrix* $\mathcal{W}_0^{(N)}$ *defined in (17) is diagonalized by the eigenvectors*

$$\psi_{n[i]}^{(N)} = \bigotimes_{l=1}^{N} \left(\exp\left(\epsilon_l^{[i]}\alpha + \epsilon_l^{[i]} \sum_{k=1}^{l-1} \epsilon_k^{[i]}\phi/2 + \mathrm{i}\theta \right) f_+^l + f_-^l \right). \tag{23}$$

The corresponding eigenvalues, ordered by the integer $n = 0, 1, \ldots, N$, *are given by*

$$\lambda_n^{(N)} = \cosh(\alpha + (N - 2n)\phi/2) \qquad \textit{where} \quad n = \left(N - \sum_{k=1}^{N} \epsilon_k^{[i]} \right) \bigg/ 2. \tag{24}$$

Proof. According to previous results, the assertion is true for $N = 1$, $N = 2$. To show it for all N, we proceed by recursion. Suppose the assertion holds for N fixed i.e. $\mathcal{W}_0^{(N)}\psi_{n[i]}^{(N)} = \lambda_n^{(N)}\psi_{n[i]}^{(N)}$ for a given set $\epsilon_j^{[i]} = \pm 1$. Replacing $N \to N + 1$ in definition (17) and using

$$\sigma_\pm f_\mp^{N+1} = f_\pm^{N+1}, \qquad \sigma_\pm f_\pm^{N+1} = 0, \qquad \sigma_3 f_\pm^{N+1} = \pm f_\pm^{N+1},$$

we obtain

$$\begin{aligned}
\mathcal{W}_0^{(N+1)} &\left(\exp\left(\epsilon_{N+1}^{[i]}\alpha + \epsilon_{N+1}^{[i]} \sum_{k=1}^{N} \epsilon_k^{[i]}\phi/2 + \mathrm{i}\theta \right) f_+^{N+1} + f_-^{N+1} \right) \otimes \psi_{n[i]}^{(N)} \\
&= \left(\left(k_+ + q^{1/2}\lambda_n^{(N)} \exp\left(\epsilon_{N+1}^{[i]}\alpha + \epsilon_{N+1}^{[i]} \sum_{k=1}^{N} \epsilon_k^{[i]}\phi/2 + \mathrm{i}\theta \right) \right) f_+^{N+1} \right. \\
&\quad \left. + \left(k_- \exp\left(\epsilon_{N+1}^{[i]}\alpha + \epsilon_{N+1}^{[i]} \sum_{k=1}^{N} \epsilon_k^{[i]}\phi/2 + \mathrm{i}\theta \right) + q^{-1/2}\lambda_n^{(N)} \right) f_-^{N+1} \right) \otimes \psi_{n[i]}^{(N)}.
\end{aligned}$$

From the definition of $k_\pm$ together with (24) and the explicit expression of n, the rhs of the equation above reduces to

$$\begin{aligned}
&\cosh\left(\alpha + \left(\sum_{k=1}^{N} \epsilon_k^{[i]} + \epsilon_{N+1}^{[i]} \right)\phi/2 \right) \\
&\qquad \times \left(\exp\left(\epsilon_{N+1}^{[i]}\alpha + \epsilon_{N+1}^{[i]} \sum_{k=1}^{N} \epsilon_k^{[i]}\phi/2 + \mathrm{i}\theta \right) f_+^{N+1} + f_-^{N+1} \right) \otimes \psi_{n[i]}^{(N)},
\end{aligned} \tag{25}$$

provided $\epsilon_{N+1}^{[i]} = \pm 1$. Identifying $\sum_{k=1}^{N+1} \epsilon_k^{[i]} = N + 1 - 2n$, the eigenvalues take the form proposed in (24) replacing N by $N + 1$. As a consequence, the eigenvectors and eigenvalues structure for N also hold for $N + 1$. The assertion being true for $N = 1, 2$, it holds for all N. □

Corollary 3.4. *Let* $\{\alpha, \alpha^*\} \in \mathbb{C}^*, \theta \in \mathbb{R}$ *be generic scalars and* ϕ *purely imaginary. The matrix* $\mathcal{W}_0^{(N)}$ *has exactly* $N + 1$ *mutually distinct eigenvalues* $\lambda_n^{(N)}$. *The dimension of an eigenspace* V_n *is*

$$\dim(V_n) = \binom{N}{n}. \tag{26}$$

Proof. First, according to the definition of n in (24) and $\epsilon_k^{[i]}=\pm 1$, one has $-N \leqslant \sum_{k=1}^N \epsilon_k^{[i]} \leqslant N$. It yields $0 \leqslant n \leqslant N$, which shows the first part of the assertion. To show (26), we proceed by recursion. For $N=1, N=2$ the assertion is true. Let N be fixed and suppose (26). Notice that $\dim(V_n)$ is the total number of configurations with index $[i]$ characterized by the sequence $\left(\epsilon_1^{[i]}=\pm 1,\ldots,\epsilon_N^{[i]}=\pm 1\right)$. Replacing N by $N+1$ in (23) and setting $n=\left(N+1-\sum_{k=1}^{N+1}\epsilon_k^{[i]}\right)/2$ there are two families of configurations associated with n:

$$\Big(\underbrace{\left(\epsilon_1^{[i]}=\pm 1,\ldots,\epsilon_j^{[i]}=\pm 1,\epsilon_{j+1}^{[i]}=\mp 1,\ldots,\epsilon_N^{[i+1]}=\pm 1\right)}_{\binom{N}{n-1}\ \text{configurations}},\epsilon_{N+1}^{[i]}=+1\Big) \qquad \text{and}$$

$$\Big(\underbrace{\left(\epsilon_1^{[i]}=\pm 1,\ldots,\epsilon_k^{[i]}=\pm 1,\epsilon_{k+1}^{[i]}=\pm 1,\ldots,\epsilon_N^{[i+1]}=\pm 1\right)}_{\binom{N}{n}\ \text{configurations}},\epsilon_{N+1}^{[i]}=-1\Big). \tag{27}$$

It follows that V_n for $N+1$ is such that $\dim(V_n)=\binom{N}{n-1}+\binom{N}{n}=\binom{N+1}{n}$ in agreement with (26). Then, (26) holds for all values of N. □

For *generic* scalars $\{\alpha,\alpha^*,\phi,\theta\}$, the eigenvectors (23) are linearly independent. In addition, $\sum_{n=0}^N \dim(V_n)=2^N=\dim(V)$ so they provide a complete basis for V. Let us now show that $\mathcal{W}_1^{(N)}$ exhibits a block tridiagonal structure in the basis $\psi_{n[i]}^{(N)}, i=1,2,\ldots,\binom{N}{n}$.

Proposition 3.5. *Define $k_+=(k_-)^\dagger=-(q^{1/2}-q^{-1/2})\,\mathrm{e}^{\mathrm{i}\theta}/2$. Let $\{\alpha,\alpha^*\}\in\mathbb{C}^*, \theta\in\mathbb{R}$ be generic scalars and ϕ purely imaginary. Let $\left\{\psi_{n[j]}^{(N)}\right\}, j=1,2,\ldots,\binom{N}{n}$ as defined by (23) the basis such that*

$$\mathcal{W}_0^{(N)}\psi_{n[j]}^{(N)}=\lambda_n^{(N)}\psi_{n[j]}^{(N)} \qquad \text{with} \quad \lambda_n^{(N)}=\cosh(\alpha+(N-2n)\phi/2) \tag{28}$$

for $n=0,1,\ldots,N$. Then, the matrix $\mathcal{W}_1^{(N)}$ acts on V_n as

$$\mathcal{W}_1^{(N)}\psi_{n[j]}^{(N)}=\sum_{i=1}^{\binom{N}{n+1}} b_{n[ij]}^{(N)}\psi_{n+1[i]}^{(N)}+\sum_{i=1}^{\binom{N}{n}} a_{n[ij]}^{(N)}\psi_{n[i]}^{(N)}+\sum_{i=1}^{\binom{N}{n-1}} c_{n[ij]}^{(N)}\psi_{n-1[i]}^{(N)}, \tag{29}$$

where the coefficients $a_{n[ij]}^{(N)}, b_{n[ij]}^{(N)}, c_{n[ij]}^{(N)}$ are determined recursively from (19).

Proof. For $N=1,2$, the assertion holds, with the entries given by (19) and (22), respectively. To show the assertion for all N, we proceed by recursion. Let N be fixed and suppose (29) is true. As noted above, given n there are two families of possible configurations, given by (27). Replacing N by $N+1$ in (23) the corresponding family of eigenvectors reads

$$\psi_{n[j]}^{(N+1)}=\left(\exp(\alpha+(N-2n)\phi/2+\mathrm{i}\theta)f_+^{N+1}+f_-^{N+1}\right)\otimes\psi_{n[j]}^{(N)}$$
$$\text{for}\quad j\in\left\{1,\ldots,\binom{N}{n}\right\},$$
$$\psi_{n[j]}^{(N+1)}=\left(\exp(-\alpha-(N-2n+2)\phi/2+\mathrm{i}\theta)f_+^{N+1}+f_-^{N+1}\right)\otimes\psi_{n-1[j-\binom{N}{n}]}^{(N)}$$
$$\text{for}\quad j\in\left\{\binom{N}{n}+1,\ldots,\binom{N+1}{n}\right\}.$$

Having these equations, we can now consider the action of $\mathcal{W}_1^{(N+1)}$ on the eigenvectors, using (17). Let us first consider the domain $j\in\left\{1,\ldots,\binom{N}{n}\right\}$:

$$\mathcal{W}_1^{(N+1)}\psi_{n[j]}^{(N+1)}=\Big(\left(k_++a_{n[jj]}^{(N)}q^{-1/2}\exp(\alpha+(N-2n)\phi/2+\mathrm{i}\theta)\right)f_+^{N+1}$$
$$+\left(k_-\exp(\alpha+(N-2n)\phi/2+\mathrm{i}\theta)+q^{1/2}a_{n[jj]}^{(N)}\right)f_-^{N+1}\Big)\otimes\psi_{n[j]}^{(N)}$$

$$
\begin{aligned}
&+\sum_{i=1,i\neq j}^{\binom{N}{n}}\Big(\big(a_{n[ij]}^{(N)}q^{-1/2}\exp(\alpha+(N-2n)\phi/2+\mathrm{i}\theta)\big)f_+^{N+1}\\
&+\big(a_{n[ij]}^{(N)}q^{1/2}\big)f_-^{N+1}\Big)\otimes\psi_{n[i]}^{(N)}\\
&+\sum_{i=1}^{\binom{N}{n+1}}\Big(\big(b_{n[ij]}^{(N)}q^{-1/2}\exp(\alpha+(N-2n)\phi/2+\mathrm{i}\theta)\big)f_+^{N+1}\\
&+\big(b_{n[ij]}^{(N)}q^{1/2}\big)f_-^{N+1}\Big)\otimes\psi_{n+1[i]}^{(N)}\\
&+\sum_{i=1}^{\binom{N}{n-1}}\Big(\big(c_{n[ij]}^{(N)}q^{-1/2}\exp(\alpha+(N-2n)\phi/2+\mathrm{i}\theta)\big)f_+^{N+1}\\
&+\big(c_{n[ij]}^{(N)}q^{1/2}\big)f_-^{N+1}\Big)\otimes\psi_{n-1[i]}^{(N)}.
\end{aligned}
\tag{30}
$$

On the other hand, $\mathcal{W}_1^{(N+1)}$ is obtained replacing N by $N+1$ in (29). It must coincide with

$$
\mathcal{W}_1^{(N+1)}\psi_{n[j]}^{(N+1)}=\sum_{i=1}^{\binom{N+1}{n+1}}b_{n[ij]}^{(N+1)}\psi_{n+1[i]}^{(N)}+\sum_{i=1}^{\binom{N+1}{n}}a_{n[ij]}^{(N+1)}\psi_{n[i]}^{(N+1)}+\sum_{i=1}^{\binom{N+1}{n-1}}c_{n[ij]}^{(N+1)}\psi_{n-1[i]}^{(N+1)}.
\tag{31}
$$

Taking into account the two families of eigenvectors and changing appropriately the indices, (31) becomes

$$
\begin{aligned}
\mathcal{W}_1^{(N+1)}\psi_{n[j]}^{(N+1)}=&\sum_{i=1}^{\binom{N}{n}}\Big(\big(a_{n[ij]}^{(N+1)}\exp(\alpha+(N-2n)\phi/2+\mathrm{i}\theta)+b_{n[i+\binom{N}{n+1})j]}^{(N+1)}\\
&\times\exp(-\alpha-(N-2n)\phi/2+\mathrm{i}\theta)\big)f_+^{N+1}+\big(a_{n[ij]}^{(N+1)}+b_{n[i+\binom{N}{n+1})j]}^{(N+1)}\big)f_-^{N+1}\Big)\otimes\psi_{n[i]}^{(N)}\\
&+\sum_{i=1}^{\binom{N}{n+1}}\Big(\big(b_{n[ij]}^{(N+1)}\exp(\alpha+(N-2n-2)\phi/2+\mathrm{i}\theta)\big)f_+^{N+1}+\big(b_{n[ij]}^{(N+1)}\big)f_-^{N+1}\Big)\otimes\psi_{n+1[i]}^{(N)}\\
&+\sum_{i=1}^{\binom{N}{n-1}}\Big(\big(a_{n[i+\binom{N}{n})j]}^{(N+1)}\exp(-\alpha-(N-2n+2)\phi/2+\mathrm{i}\theta)+c_{n[ij]}^{(N+1)}\\
&\times\exp(\alpha+(N-2n+2)\phi/2+\mathrm{i}\theta)\big)f_+^{N+1}+\big(a_{n[i+\binom{N}{n})j]}^{(N+1)}+c_{n[ij]}^{(N+1)}\big)f_-^{N+1}\Big)\otimes\psi_{n-1[i]}^{(N)}\\
&+\sum_{i=1}^{\binom{N}{n-2}}\Big(c_{n[i+\binom{N}{n-1})j]}^{(N+1)}\exp(-\alpha-(N-2n+4)\phi/2+\mathrm{i}\theta)\big)f_+^{N+1}\\
&+\big(c_{n[i+\binom{N}{n-1})j]}^{(N+1)}\big)f_-^{N+1}\Big)\otimes\psi_{n-2[i]}^{(N)}.
\end{aligned}
\tag{32}
$$

Identifying (30) with (32), the $N+1$-th entries are determined recursively in terms of the N-th entries (see appendix A) and the final result is consistent with the block tridiagonal structure. In particular,

$$
c_{n[i+\binom{N-1}{n})j]}^{(N+1)}\equiv 0\qquad\text{for}\qquad i\in\left\{1,\ldots,\binom{N}{n-2}\right\},\qquad j\in\left\{1,\ldots,\binom{N}{n}\right\}.
$$

Then, (29) is satisfied for $N\to N+1$ and $j\in\left\{1,\ldots,\binom{N}{n}\right\}$, provided it holds for N. We now consider the action of $\mathcal{W}_1^{(N+1)}$ on the eigenvectors in the domain $j\in\left\{\binom{N}{n}+1,\ldots,\binom{N+1}{n}\right\}$.

We follow the same approach as above: after a straightforward calculation, we find similarly that (29) for $N \rightarrow N+1$ is also satisfied, and in particular

$$b^{(N+1)}_{n[ij]} \equiv 0 \qquad \text{for} \quad i \in \left\{1, \ldots, \binom{N}{n+1}\right\}, \qquad j \in \left\{\binom{N}{n}+1, \ldots, \binom{N+1}{n}\right\}.$$

As a consequence, (29) holds for any $j \in \left\{\left(1, \ldots, \binom{N+1}{n}\right)\right.$. We have checked (29) for $N = 1, 2$ and $N = 3$—the case where some entries are vanishing—so we conclude that the assertion holds for all N. Explicit expressions for the entries are reported in appendix A. □

Definition 3.6. *Let V denote the finite-dimensional representation (of dimension 2^N) on which $\mathcal{W}_1^{(N)}$ acts. Suppose $\mathcal{W}_1^{(N)}$ is diagonalizable on V, and let $V_0^*, V_1^*, V_2^*, \ldots$ denote a standard ordering of the eigenspaces of $\mathcal{W}_1^{(N)}$. For $s = 0, 1, 2, \ldots$, let $\tilde{\lambda}_s^{(N)}$ denote the eigenvalue sequence of $\mathcal{W}_1^{(N)}$.*

Proposition 3.7. *Define $k_+ = (k_-)^\dagger = -(q^{1/2} - q^{-1/2})\, \mathrm{e}^{\mathrm{i}\theta}/2$. Let $\{\alpha, \alpha^*\} \in \mathbb{C}^*, \theta \in \mathbb{R}$ be generic scalars and ϕ purely imaginary. Introduce the parameters $\tilde{\epsilon}_l^{[k]} = \pm 1, l = 1, \ldots, N, k \in \left\{1, \ldots, \binom{N}{s}\right\}$. For any N, the set $\varphi^{(N)}_{s[k]}$ defined by*

$$\varphi^{(N)}_{s[k]} = \bigotimes_{l=1}^{N} \left(-\exp\left(-\tilde{\epsilon}_l^{[k]}\alpha^* - \tilde{\epsilon}_l^{[k]} \sum_{j=1}^{l-1} \tilde{\epsilon}_j^{[k]} \phi/2 + \mathrm{i}\theta \right) f_+^l + f_-^l \right),$$
$$s = \left(N - \sum_{j=1}^{N} \tilde{\epsilon}_j^{[k]} \right) \Big/ 2 \tag{33}$$

forms a complete basis of V. In this basis, the matrices $\mathcal{W}_0^{(N)}, \mathcal{W}_1^{(N)}$ are such that

$$\mathcal{W}_1^{(N)} \varphi^{(N)}_{s[k]} = \tilde{\lambda}_s^{(N)} \varphi^{(N)}_{s[k]} \quad \textit{with} \quad \tilde{\lambda}_s^{(N)} = \cosh(\alpha^* + (N-2s)\phi/2),$$
$$\mathcal{W}_0^{(N)} \varphi^{(N)}_{s[k]} = \sum_{l=1}^{\binom{N}{s+1}} \tilde{b}^{(N)}_{s[lk]} \varphi^{(N)}_{s+1[l]} + \sum_{l=1}^{\binom{N}{s}} \tilde{a}^{(N)}_{s[lk]} \varphi^{(N)}_{s[l]} + \sum_{l=1}^{\binom{N}{s-1}} \tilde{c}^{(N)}_{s[lk]} \varphi^{(N)}_{s-1[l]}, \tag{34}$$

where the coefficients $\tilde{a}^{(N)}_{s[lk]}, \tilde{b}^{(N)}_{s[lk]}, \tilde{c}^{(N)}_{s[lk]}$ are obtained from appendix A with the substitution (20).

Proof. According to expressions (17) and the previous analysis, the proof is straightforward. □

Lemma 3.8. *Let $\{\alpha, \alpha^*\} \in \mathbb{C}^*, \theta \in \mathbb{R}$ be generic scalars and ϕ purely imaginary. For any N, let the matrices $\mathcal{W}_0^{(N)}, \mathcal{W}_1^{(N)}$ be defined by (17). Then, $\mathcal{W}_0^{(N)}, \mathcal{W}_1^{(N)}$ act on V as a TD pair of diameter N.*

Proof. We show that conditions 1–4 of definition 2.1 are satisfied. According to propositions 3.3 and 3.7, condition 1 is satisfied. Having $n \in \{0, 1, \ldots, N\}$ and the block tridiagonal structure (29), (34), $V = \bigoplus_{n=0}^{N} V_n$ is irreducible i.e. condition 4 is satisfied. Finally, propositions 3.5 and 3.7 imply that conditions 2 and 3 are satisfied. Then, $\mathcal{W}_0^{(N)}, \mathcal{W}_1^{(N)}$ is a TD pair. From corollary 3.4 and proposition 3.7 one finds that the diameter of the pair is $d = N$. □

Remark 3.9. In agreement with [9], the eigenvalue sequence $\left(\lambda_0^{(N)}, \lambda_1^{(N)}, \ldots, \lambda_N^{(N)}\right)$ and dual eigenvalue sequence $\left(\tilde{\lambda}_0^{(N)}, \tilde{\lambda}_1^{(N)}, \ldots, \tilde{\lambda}_N^{(N)}\right)$ satisfy

$$\frac{\lambda_{n-2}^{(N)} - \lambda_{n+1}^{(N)}}{\lambda_{n-1}^{(N)} - \lambda_{n}^{(N)}} = q + q^{-1} + 1 \qquad \text{and} \qquad \frac{\tilde{\lambda}_{s-2}^{(N)} - \tilde{\lambda}_{s+1}^{(N)}}{\tilde{\lambda}_{s-1}^{(N)} - \tilde{\lambda}_{s}^{(N)}} = q + q^{-1} + 1$$

for $n, s = 2, \ldots, N-1$. The parameters ρ, ρ^* in (5) are given by $\rho = \rho^* = -(q - q^{-1})^2/4$.

4. Examples of related symmetric functions

As suggested in [10], given the connection between Leonard pairs[4] and the Askey–Wilson orthogonal polynomials (q-Racah, q-Hahn, ...) on a discrete support it is interesting to investigate the family of functions associated with more general TD pairs, which is the purpose of this section. As shown in [5], Askey–Wilson polynomials of general form with a discrete argument satisfy a 3-term recurrence relation and a second-order q-difference equation that can be derived starting from the structure of the finite-dimensional representations of the Askey–Wilson algebra. In particular, to the four parameters of the polynomials correspond the four independent structure constants of the algebra. Following a similar point of view, for the TD pairs (15) it is possible to deduce a coupled system of recurrence relations and a set of second-order q-difference equations. Below, we consider in particular the family (17) for α, α^* purely imaginary[5] so that $\{\lambda_n^{(N)}, \tilde{\lambda}_s^{(N)}\} \in \mathbb{R}$. For this choice, $\left(\mathcal{W}_0^{(N)}\right)^\dagger \equiv \mathcal{W}_1^{(N)}\big|_{\alpha^* \to \alpha}$ for any N. So, we define $\tilde{\varphi}_{s[k]}^{(N)} \equiv \psi_{n[k]}^{(N)}\big|_{\alpha \to \alpha^*, n \to s}$.

4.1. Example $N = 1$

For $N = 1$, according to the previous analysis the pair $\mathcal{W}_0^{(1)}, \mathcal{W}_1^{(1)}$ is a Leonard pair. By analogy with [5], using the scalar product in the canonical basis $f_\pm^{(1)}$ we introduce the function $\mathcal{F}_{n[1]}^{(1)}(\tilde{\lambda})$ for $n = 0, 1$ such that

$$\left\langle \tilde{\varphi}_{s[1]}^{(1)}, \psi_{n[1]}^{(1)} \right\rangle = U_1^{(1)}(s) \mathcal{F}_{n[1]}^{(1)}\left(\tilde{\lambda}_s^{(1)}\right) \quad \text{with} \quad \mathcal{F}_{0[1]}^{(1)}\left(\tilde{\lambda}_s^{(1)}\right) \equiv 1 \tag{35}$$

for all $s = 0, 1$. For this choice of normalization, the coefficient $U_1^{(1)}(s) = \mathrm{e}^{\alpha - (1-2s)\alpha^*} + 1$, as follows from definitions (18). Using the matrix representation (19) for $\mathcal{W}_1^{(1)}$, the expression $\left\langle \tilde{\varphi}_{s[1]}^{(1)}, \mathcal{W}_1^{(1)} \psi_{n[1]}^{(1)} \right\rangle$ yields the following (2-term) recurrence relations:

$$\begin{aligned} \tilde{\lambda}_s^{(1)} \mathcal{F}_{0[1]}^{(1)}\left(\tilde{\lambda}_s^{(1)}\right) &= b_0^{(1)} \mathcal{F}_{1[1]}^{(1)}\left(\tilde{\lambda}_s^{(1)}\right) + a_0^{(1)} \mathcal{F}_{0[1]}^{(1)}\left(\tilde{\lambda}_s^{(1)}\right), \\ \tilde{\lambda}_s^{(1)} \mathcal{F}_{1[1]}^{(1)}\left(\tilde{\lambda}_s^{(1)}\right) &= a_1^{(1)} \mathcal{F}_{0[1]}^{(1)}\left(\tilde{\lambda}_s^{(1)}\right) + c_1^{(1)} \mathcal{F}_{1[1]}^{(1)}\left(\tilde{\lambda}_s^{(1)}\right) \end{aligned} \tag{36}$$

for any $s = 0, 1$. For the choice (35), the first equation determines uniquely $\mathcal{F}_{1[1]}^{(1)}(\tilde{\lambda}) = \left(\tilde{\lambda} - a_0^{(1)}\right)\big/ b_0^{(1)}$, whose form coincides with the first of the Askey–Wilson polynomials with argument $\tilde{\lambda}$. Replacing this latter expression in the second equation of (37), the values of $\tilde{\lambda}$ are restricted to $\tilde{\lambda} = \cosh(\alpha^* \pm \phi/2)$, in perfect agreement with the structure of the eigenvalues in the dual basis as found in section 3.1.

[4] Leonard pairs are the simplest examples of tridiagonal pairs. Their shape vector being $(1, \ldots, 1)$, all eigenspaces have dimension one. In particular, in the basis that diagonalizes A (resp. A^*), the matrix representing A^* (resp. A) is irreducible tridiagonal.

[5] A similar analysis can be done for $\alpha^\dagger = \alpha^*$ in which case $(\mathcal{W}_0^{(N)})^\dagger \equiv \mathcal{W}_1^{(N)}$.

On the other hand, we may expand the expression $\langle \tilde{\varphi}^{(1)}_{s[1]}, \mathcal{W}^{(1)}_0 \psi^{(1)}_{n[1]} \rangle$ using definition (35). In this case, we now get the following (first order) q-difference equations:

$$\begin{aligned}
\lambda^{(1)}_n \mathcal{F}^{(1)}_{n[1]}\big(\tilde{\lambda}^{(1)}_0\big) &= \tilde{b}^{(1)}_0 \frac{U^{(1)}_1(1)}{U^{(1)}_1(0)} \mathcal{F}^{(1)}_{n[1]}\big(\tilde{\lambda}^{(1)}_1\big) + \tilde{a}^{(1)}_0 \mathcal{F}^{(1)}_{n[1]}\big(\tilde{\lambda}^{(1)}_0\big), \\
\lambda^{(1)}_n \mathcal{F}^{(1)}_{n[1]}\big(\tilde{\lambda}^{(1)}_1\big) &= \tilde{a}^{(1)}_1 \mathcal{F}^{(1)}_{n[1]}\big(\tilde{\lambda}^{(1)}_1\big) + \tilde{c}^{(1)}_1 \frac{U^{(1)}_1(0)}{U^{(1)}_1(1)} \mathcal{F}^{(1)}_{n[1]}\big(\tilde{\lambda}^{(1)}_0\big)
\end{aligned} \tag{37}$$

for any $n = 0, 1$. The fact that we obtain a 2-term recurrence relation and a first-order q-difference equation is not surprising, as we started with spin-$j = 1/2$ representations of $U_{q^{1/2}}(sl_2)$. Indeed, 3-term recurrence relations and second-order q-difference equations appear for spin $-j > 1/2$ representations of $U_{q^{1/2}}(sl_2)$ only.

4.2. *Example $N = 2$*

For $N = 2$, the pair $\mathcal{W}^{(2)}_0, \mathcal{W}^{(2)}_1$ is the first example of TD pair that is not a Leonard pair, due to the multiplicity of one of the eigenvalues. As before, we introduce the function $\mathcal{F}^{(2)}_{n[i]}(\tilde{\lambda})$ for any $n = 0, 1, 2$ and $i \in \left\{\left(1, \ldots, \binom{2}{n}\right)\right\}$ such that

$$\big\langle \tilde{\varphi}^{(2)}_{s[k]}, \psi^{(2)}_{n[i]} \big\rangle = U^{(2)}_k(s) \mathcal{F}^{(2)}_{n[i]}\big(\tilde{\lambda}^{(2)}_s\big) \tag{38}$$

for $s = 0, 1, 2$ and $k \in \left\{\left(1, \ldots, \binom{2}{s}\right)\right\}$. Choosing the normalization

$$\mathcal{F}^{(2)}_{0[1]}\big(\tilde{\lambda}^{(2)}_s\big) \equiv 1 \qquad \text{for any} \quad s = 0, 1, 2, \tag{39}$$

and using the scalar product in the canonical basis and the explicit form of the eigenvectors (21), it follows

$$\begin{aligned}
&U^{(2)}_1(s=0) = (\mathrm{e}^{\alpha-\alpha^*}+1)^2, && U^{(2)}_1(s=1) = (\mathrm{e}^{\alpha-\alpha^*+\phi}+1)(\mathrm{e}^{\alpha+\alpha^*}+1),\\
&U^{(2)}_2(s=1) = (\mathrm{e}^{\alpha+\alpha^*+\phi}+1)(\mathrm{e}^{\alpha-\alpha^*}+1), && U^{(2)}_1(s=2) = (\mathrm{e}^{\alpha+\alpha^*}+1)^2.
\end{aligned}$$

Then, using the matrix representation (22) for $\mathcal{W}^{(2)}_1$ we immediately derive the following (coupled) system of recurrence relations

$$\tilde{\lambda}^{(2)}_s \mathcal{F}^{(2)}_{0[1]}\big(\tilde{\lambda}^{(2)}_s\big) = b^{(2)}_{0[11]}\mathcal{F}^{(2)}_{1[1]}\big(\tilde{\lambda}^{(2)}_s\big) + b^{(2)}_{0[21]}\mathcal{F}^{(2)}_{1[2]}\big(\tilde{\lambda}^{(2)}_s\big) + a^{(2)}_{0[11]}\mathcal{F}^{(2)}_{0[1]}\big(\tilde{\lambda}^{(2)}_s\big), \tag{40}$$

$$\tilde{\lambda}^{(2)}_s \mathcal{F}^{(2)}_{1[1]}\big(\tilde{\lambda}^{(2)}_s\big) = b^{(2)}_{1[11]}\mathcal{F}^{(2)}_{2[1]}\big(\tilde{\lambda}^{(2)}_s\big) + a^{(2)}_{1[11]}\mathcal{F}^{(2)}_{1[1]}\big(\tilde{\lambda}^{(2)}_s\big) + a^{(2)}_{1[21]}\mathcal{F}^{(2)}_{1[2]}\big(\tilde{\lambda}^{(2)}_s\big) + c^{(2)}_{1[11]}\mathcal{F}^{(2)}_{0[1]}\big(\tilde{\lambda}^{(2)}_s\big), \tag{41}$$

$$\tilde{\lambda}^{(2)}_s \mathcal{F}^{(2)}_{1[2]}\big(\tilde{\lambda}^{(2)}_s\big) = b^{(2)}_{1[12]}\mathcal{F}^{(2)}_{2[1]}\big(\tilde{\lambda}^{(2)}_s\big) + a^{(2)}_{1[12]}\mathcal{F}^{(2)}_{1[1]}\big(\tilde{\lambda}^{(2)}_s\big) + a^{(2)}_{1[22]}\mathcal{F}^{(2)}_{1[2]}\big(\tilde{\lambda}^{(2)}_s\big) + c^{(2)}_{1[12]}\mathcal{F}^{(2)}_{0[1]}\big(\tilde{\lambda}^{(2)}_s\big), \tag{42}$$

$$\tilde{\lambda}^{(2)}_s \mathcal{F}^{(2)}_{2[1]}\big(\tilde{\lambda}^{(2)}_s\big) = a^{(2)}_{2[11]}\mathcal{F}^{(2)}_{2[1]}\big(\tilde{\lambda}^{(2)}_s\big) + c^{(2)}_{2[11]}\mathcal{F}^{(2)}_{1[1]}\big(\tilde{\lambda}^{(2)}_s\big) + c^{(2)}_{2[21]}\mathcal{F}^{(2)}_{1[2]}\big(\tilde{\lambda}^{(2)}_s\big), \tag{43}$$

for any $s = 0, 1, 2$. For the normalization (39), a straightforward calculation shows that $\mathcal{F}^{(2)}_{n[i]}(\tilde{\lambda})$ are *rational* functions of $\tilde{\lambda}$ uniquely determined by equations (40), (41) and (43) for any values of $\tilde{\lambda}$. Explicitly, together with (39) they read

$$\begin{aligned}
\mathcal{F}^{(2)}_{1[1]}(\tilde{\lambda}) &= -\frac{\mathrm{e}^{-\alpha-\phi/2}\sinh\phi(\cosh\alpha+\cosh\alpha^*)}{\sinh(\alpha-\phi/2)\cosh\alpha^*+\cosh(\alpha+\phi/2)\sinh\alpha-\sinh(\phi/2)}\frac{(\tilde{\lambda}-u_{1[1]})}{(\tilde{\lambda}-v)},\\
\mathcal{F}^{(2)}_{1[2]}(\tilde{\lambda}) &= \frac{\mathrm{e}^{-\alpha-\phi/2}\sinh(\alpha+\phi/2)\sinh\alpha}{\sinh(\phi/2)(\sinh(\alpha-\phi/2)\cosh\alpha^*+\cosh(\alpha+\phi/2)\sinh\alpha-\sinh(\phi/2))}\\
&\quad\times\frac{\big(\tilde{\lambda}-u^{(+)}_{1[2]}\big)\big(\tilde{\lambda}-u^{(-)}_{1[2]}\big)}{(\tilde{\lambda}-v)},\\
\mathcal{F}^{(2)}_{2[1]}(\tilde{\lambda}) &= -\frac{\mathrm{e}^{-2\alpha}\sinh(\alpha+\phi/2)(\cosh\alpha+\cosh\alpha^*)}{\sinh(\alpha-\phi/2)\cosh\alpha^*+\cosh(\alpha+\phi/2)\sinh\alpha-\sinh(\phi/2)}\frac{(\tilde{\lambda}-u_{2[1]})}{(\tilde{\lambda}-v)},
\end{aligned} \tag{44}$$

where the expressions $u_{1[1]}, u^{(\pm)}_{1[2]}, u_{2[1]}$ and v are reported in appendix B. Note that these expressions are invariant under the substitution $\alpha^* \rightarrow -\alpha^*$. Finally, equation (42) fixes the values of $\tilde{\lambda}$: replacing (44) in (42) and using the explicit expressions of the coefficients, one gets a third-order polynomial in $\tilde{\lambda}$, whose roots reproduce exactly the eigenvalue sequence (24) for $N = 2$, as expected.

On the other hand, it is possible to expand $\langle \tilde{\varphi}^{(2)}_{s[k]}, \mathcal{W}^{(2)}_0 \psi^{(2)}_{n[i]} \rangle$ using (38) which leads to a set of q-difference equations. This being analogous to that for higher values of N, we refer the reader to the analysis below.

4.3. Generalized recurrence relations and q-difference equations

Let us now turn to general values of N. Due to the block tridiagonal structure of the matrices $\mathcal{W}^{(N)}_0, \mathcal{W}^{(N)}_1$ with respect to their dual basis, for arbitrary N it is easy to derive a system of recurrence relations and q-difference equations. By analogy with (38), we introduce the functions $\mathcal{F}^{(N)}_{n[i]}(\tilde{\lambda})$ for any $n = 0, 1, \ldots, N$ and $i \in \left\{\left(1, \ldots, \binom{N}{n}\right)\right\}$:

$$\left\langle \tilde{\varphi}^{(N)}_{s[k]}, \psi^{(N)}_{n[i]} \right\rangle = U^{(N)}_k(s)\mathcal{F}^{(N)}_{n[i]}\left(\tilde{\lambda}^{(N)}_s\right) \qquad \text{with} \quad \mathcal{F}^{(N)}_{0[1]}\left(\tilde{\lambda}^{(N)}_s\right) \equiv 1 \tag{45}$$

for any $s = 0, 1, \ldots, N$ and $k \in \left\{\left(1, \ldots, \binom{N}{s}\right)\right\}$. Here, the coefficients $U^{(N)}_k(s)$ can be calculated recursively on the basis of previous results: using the scalar product in the canonical basis together with the explicit form of the eigenvectors (23) for $s = 0, 1, \ldots, N$ we find:

$$\begin{aligned}
U^{(N)}_k(s) &= \left(e^{(\alpha-\alpha^*+s\phi)} + 1\right)U^{(N-1)}_k(s) && \text{for} \quad k \in \left\{1, \ldots, \binom{N-1}{s}\right\}, \\
U^{(N)}_k(s) &= \left(e^{(\alpha+\alpha^*+(N-s)\phi)} + 1\right)U^{(N-1)}_{k-\binom{N-1}{s}}(s-1) && \text{for} \quad k \in \left\{\binom{N-1}{s} + 1, \ldots, \binom{N}{s}\right\}.
\end{aligned} \tag{46}$$

First, let us focus on the system of recurrence relations for the rational functions $\mathcal{F}^{(N)}_{n[i]}(\tilde{\lambda})$. Using the results of previous sections, from $\langle \tilde{\varphi}^{(N)}_{s[k]}, \mathcal{W}^{(N)}_1 \psi^{(N)}_{n[i]} \rangle$ we immediately get

$$\tilde{\lambda}\mathcal{F}^{(N)}_{n[j]}(\tilde{\lambda}) = \sum_{i=1}^{\binom{N}{n+1}} b^{(N)}_{n[ij]}\mathcal{F}^{(N)}_{n+1[i]}(\tilde{\lambda}) + \sum_{i=1}^{\binom{N}{n}} a^{(N)}_{n[ij]}\mathcal{F}^{(N)}_{n[i]}(\tilde{\lambda}) + \sum_{i=1}^{\binom{N}{n-1}} c^{(N)}_{n[ij]}\mathcal{F}^{(N)}_{n-1[i]}(\tilde{\lambda}) \tag{47}$$

for $\tilde{\lambda} \in \left\{\tilde{\lambda}^{(N)}_s\right\}, n, s = 0, 1, \ldots, N, j \in \left\{1, \ldots, \binom{N}{n}\right\}$ with the coefficients given in appendix A. Note that similarly to the case $N = 2$, $2^N - 1$ equations will determine uniquely the explicit form of $\mathcal{F}^{(N)}_{n[i]}(\tilde{\lambda})$. The remaining equation, associated with $n = N - 1$ and $j = N$, determines the eigenvalues to be in the set $\tilde{\lambda} \in \left\{\tilde{\lambda}^{(N)}_s\right\}$, $s = 0, 1, \ldots, N$ given by (34).

To derive a system of q-difference equations for the rational functions $\mathcal{F}^{(N)}_{n[i]}(\tilde{\lambda})$, we start from $\langle \tilde{\varphi}^{(N)}_{s[k]}, \mathcal{W}^{(N)}_0 \psi^{(N)}_{n[i]} \rangle$. Let us introduce the discrete q-difference operator $\eta^{\pm 1}g(\tilde{\lambda}_s) = g(\tilde{\lambda}_{s\pm 1})$. Combining the normalization coefficients $U^{(N)}_k(s)$ given by (46) with the 'dual' entries we obtain a degenerate family of second-order q-difference operators such that

$$\mathbb{D}^{(N)}_k(s)\mathcal{F}^{(N)}_{n[j]}\left(\tilde{\lambda}^{(N)}_s\right) = \lambda^{(N)}_n \mathcal{F}^{(N)}_{n[j]}\left(\tilde{\lambda}^{(N)}_s\right) \tag{48}$$

for any $k = 1, \ldots, \binom{N}{s}$, where

$$\mathbb{D}^{(N)}_k(s) \equiv \Phi^{(N)}_k(s)\eta + \overline{\Phi}^{(N)}_k(s)\eta^{-1} + \mu^{(N)}_k(s),$$

$$\Phi^{(N)}_k(s) = \sum_{l=1}^{\binom{N}{s+1}} \tilde{b}^{(N)}_{s[lk]} \frac{U^{(N)}_l(s+1)}{U^{(N)}_k(s)}, \qquad \overline{\Phi}^{(N)}_k(s) = \sum_{l=1}^{\binom{N}{s-1}} \tilde{c}^{(N)}_{s[lk]} \frac{U^{(N)}_l(s-1)}{U^{(N)}_k(s)},$$

$$\mu_k^{(N)}(s) = \sum_{l=1}^{\binom{N}{s}} \tilde{a}_{s[lk]}^{(N)} \frac{U_l^{(N)}(s)}{U_k^{(N)}(s)}.$$

It should be pointed out that for $N = 1$, the operator $\mathbb{D}_1^{(1)}(s)$ is a special case (as we chose the spin-1/2 representation of $U_{q^{1/2}}(sl_2)$) of the Askey–Wilson operator derived in [5]. As a consequence, for higher values of N the $\binom{N}{s}$ q-difference operators $\mathbb{D}_k^{(N)}(s)$ can be seen as some generalizations of the Askey–Wilson operator. Although it is an interesting problem to explore some consequences in integrable systems and the links with a recent work on Bethe ansatz and q-Sturm–Liouville problems [22], we do not pursue the analysis further and leave it for a separate work [16]. Note that for $N = 2$ we have checked explicitly that the rational functions (44) satisfy the degenerate set of q-difference equations (48).

4.4. Orthogonality

The system of Askey–Wilson polynomials is orthogonal with respect to a certain discrete weight function. This can be shown, for instance, using the completeness of the two dual bases which diagonalize the elements of the Askey–Wilson algebra [5]. Such property can be extended to the system of rational functions $\mathcal{F}_{n[j]}^{(N)}(\tilde{\lambda})$ (defined on a real discrete support) obeying (47), (48), having in mind that the two bases (23) and (33) are complete for general values of the parameters. To show that, we introduce the basis of dual eigenvectors $\tilde{\varphi}_{s[k]}^{(N)} \equiv \psi_{n[k]}^{(N)}\big|_{\alpha\to\alpha^*,n\to s}$ and $\tilde{\psi}_{n[i]}^{(N)} \equiv \varphi_{n[i]}^{(N)}\big|_{\alpha^*\to\alpha,s\to n}$. It is easy to check that

$$\mathcal{N}_{n[i]}^{(N)}\big\langle\tilde{\psi}_{n[i]}^{(N)}, \psi_{m[j]}^{(N)}\big\rangle = \delta_{n,m}\delta_{i,j} \qquad \text{and} \qquad \tilde{\mathcal{N}}_{s[k]}^{(N)}\big\langle\tilde{\varphi}_{s[k]}^{(N)}, \varphi_{r[l]}^{(N)}\big\rangle = \delta_{s,r}\delta_{k,l}, \tag{49}$$

where, using the explicit expression of the eigenvectors, one can easily obtain recursively the normalization coefficients. For instance, for $n = 0, \ldots, N$ one finds

$$\begin{aligned}
\mathcal{N}_{n[i]}^{(N)} &= \mathcal{N}_{n[i]}^{(N-1)}(1 - \exp((2\alpha + (N-1-2n)\phi)))^{-1} \qquad \text{for} \quad i \in \left\{1, \ldots, \binom{N-1}{n}\right\}, \\
\mathcal{N}_{n[i]}^{(N)} &= \mathcal{N}_{n[i-\binom{N-1}{n}]}^{(N-1)}(1 - \exp((-2\alpha - (N+1-2n)\phi)))^{-1} \\
&\qquad \text{for} \quad i \in \left\{\binom{N-1}{n} + 1, \ldots, \binom{N}{n}\right\},
\end{aligned} \tag{50}$$

and similarly $\tilde{\mathcal{N}}_{s[k]}^{(N)} = \mathcal{N}_{n[i]}^{(N)}\big|_{\{n,i,\alpha,\phi\}\to\{s,k,-\alpha^*,-\phi\}}$ for $s = 0, 1, \ldots, N$. Then, the completeness of the two eigenbasis together with the definition of the normalization coefficients yield

$$\big\langle\tilde{\psi}_{n[i]}^{(N)}, \psi_{m[j]}^{(N)}\big\rangle = \sum_{s=0}^{N}\sum_{k=1}^{\binom{N}{s}} \tilde{\mathcal{N}}_{s[k]}^{(N)}\big\langle\tilde{\psi}_{n[i]}^{(N)}, \varphi_{s[k]}^{(N)}\big\rangle\big\langle\tilde{\varphi}_{s[k]}^{(N)}, \psi_{m[j]}^{(N)}\big\rangle.$$

Replacing (45) in the above equation, and using $\big\langle\tilde{\psi}_{n[i]}^{(N)}, \varphi_{s[k]}^{(N)}\big\rangle = \big\langle\tilde{\varphi}_{s[k]}^{(N)}, \psi_{m[j]}^{(N)}\big\rangle$ we find that the system of rational functions $\mathcal{F}_{n[j]}^{(N)}(\tilde{\lambda}_s)$ of the discrete argument $\tilde{\lambda}_s$ are orthogonal on the $N+1$ points on the interval of the real axis $0 \leqslant s \leqslant N$. The condition of orthogonality reads

$$\sum_{s=0}^{N} w^{(N)}(s)\mathcal{F}_{n[i]}^{(N)}(\tilde{\lambda}_s)\mathcal{F}_{m[j]}^{(N)}(\tilde{\lambda}_s) = \big(\mathcal{N}_{n[i]}^{(N)}\big)^{-1}\delta_{n,m}\delta_{i,j} \quad \text{where} \quad w^{(N)}(s) = \sum_{k=1}^{\binom{N}{s}} \tilde{\mathcal{N}}_{s[k]}^{(N)}\big(U_k^{(N)}(s)\big)^2$$

is the discrete weight function that ensures the orthogonality of the sytem. Given N, its explicit expression is derived recursively using (50) and (46). In particular, it is an exercise to check the orthogonality condition for $N = 1$ (in which case our construction reduces to a special case (spin-1/2) of [5]) and $N = 2$.

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Appendix A. Coefficients of the matrices $A_n^{(N)}, B_n^{(N)}, C_n^{(N)}$

We can organize the set of eigenvectors by eigenspaces V_n and set $\Psi_n^{(N)} = \left[\psi_{n[1]}^{(N)}, \psi_{n[2]}^{(N)}, \ldots, \psi_{n[\binom{N}{n}]}^{(N)}\right]^t$. Then, relations (28) and (29) can be written in the form

$$\mathcal{W}_0^{(N)}\Psi_n^{(N)} = \lambda_n^{(N)}\Psi_n^{(N)}, \qquad \mathcal{W}_1^{(N)}\Psi_n^{(N)} = B_n^{(N)}\Psi_{n+1}^{(N)} + A_n^{(N)}\Psi_n^{(N)} + C_n^{(N)}\Psi_{n-1}^{(N)}, \tag{A.1}$$

where $A_n^{(N)}, B_n^{(N)}, C_n^{(N)}$ are submatrices with entries $\left(X_n^{(N)}\right)_{[ji]} \equiv \left(x_n^{(N)}\right)_{[ij]}$ which have been determined recursively in section 3, with 'initial' conditions (19). The size of each submatrix is determined by the multiplicity of the eigenvalue. We obtain

- *Matrix* $A_n^{(N)}$

  For $i \in \left\{1, \ldots, \binom{N-1}{n}\right\}$:
  $$a_{n[ii]}^{(N)} = \frac{a_{n[ii]}^{(N-1)}\sinh(\alpha + (N-2-2n)\phi/2) - \sinh(\phi/2)}{\sinh(\alpha + (N-1-2n)\phi/2)},$$
  For $i \in \left\{\binom{N-1}{n} + 1, \ldots, \binom{N}{n}\right\}$:
  $$a_{n[ii]}^{(N)} = \frac{a_{n-1[i-\binom{N-1}{n}i-\binom{N-1}{n}]}^{(N-1)}\sinh(\alpha + (N+2-2n)\phi/2) + \sinh(\phi/2)}{\sinh(\alpha + (N+1-2n)\phi/2)},$$
  For $i \in \left\{1, \ldots, \binom{N-1}{n}\right\}, \quad j \in \left\{\binom{N-1}{n} + 1, \ldots, \binom{N}{n}\right\}$:
  $$a_{n[ij]}^{(N)} = -\frac{\exp(-\alpha - (N-2n)\phi/2)\sinh(\phi)}{\sinh(\alpha + (N-1-2n)\phi/2)} b_{n-1[ij]}^{(N-1)},$$
  For $i \in \left\{\binom{N-1}{n} + 1, \ldots, \binom{N}{n}\right\}, \quad j \in \left\{1, \ldots, \binom{N-1}{n}\right\}$:
  $$a_{n[ij]}^{(N)} = \frac{\exp(\alpha + (N-2n)\phi/2)\sinh(\phi)}{\sinh(\alpha + (N+1-2n)\phi/2)} c_{n[i-\binom{N-1}{n}j]}^{(N-1)},$$
  For $i, j \in \left\{1, \ldots, \binom{N-1}{n}\right\}, \quad i \neq j$:
  $$a_{n[ij]}^{(N)} = \frac{\sinh(\alpha + (N-2-2n)\phi/2)}{\sinh(\alpha + (N-1-2n)\phi/2)} a_{n[ij]}^{(N-1)},$$
  For $i, j \in \left\{\binom{N-1}{n} + 1, \ldots, \binom{N}{n}\right\}, \quad i \neq j$:
  $$a_{n[ij]}^{(N)} = \frac{\sinh(\alpha + (N+2-2n)\phi/2)}{\sinh(\alpha + (N+1-2n)\phi/2)} a_{n-1[i-\binom{N-1}{n}j-\binom{N-1}{n}]}^{(N-1)}.$$

- *Matrix* $B_n^{(N)}$:

  For $i \in \left\{1, \ldots, \binom{N-1}{n+1}\right\}, \quad j \in \left\{1, \ldots, \binom{N-1}{n}\right\}$: $\quad b_{n[ij]}^{(N)} = e^{\phi/2} b_{n[ij]}^{(N-1)},$

  For $i \in \left\{1, \ldots, \binom{N-1}{n+1}\right\}, \quad j \in \left\{\binom{N-1}{n} + 1, \ldots, \binom{N}{n}\right\}$: $\quad b_{n[ij]}^{(N)} = 0,$

For $\quad i \in \left\{\binom{N-1}{n+1}+1,\ldots,\binom{N}{n+1}\right\},\quad j \in \left\{\binom{N-1}{n}+1,\ldots,\binom{N}{n}\right\}$:

$$b^{(N)}_{n[ij]} = \exp(-\phi/2)\frac{\sinh(\alpha+(N+1-2n)\phi/2)}{\sinh(\alpha+(N-1-2n)\phi/2)}b^{(N-1)}_{n-1[i-\binom{N-1}{n+1}j]},$$

For $\quad i \in \left\{\binom{N-1}{n+1}+1,\ldots,\binom{N}{n+1}\right\}$:

$$b^{(N)}_{n[ii-\binom{N-1}{n+1}]} = \frac{\exp(\alpha+(N-1-2n)\phi/2)\sinh(\phi/2)}{\sinh(\alpha+(N-1-2n)\phi/2)} \times \left(a^{(N-1)}_{n[i-\binom{N-1}{n+1}i-\binom{N-1}{n+1}]}+\cosh(\alpha+(N-1-2n)\phi/2)\right),$$

For $\quad i \in \left\{\binom{N-1}{n+1}+1,\ldots,\binom{N}{n+1}\right\}, j \in \left\{1,\ldots,\binom{N-1}{n}\right\},\quad i \neq j+\binom{N-1}{n+1}$:

$$b^{(N)}_{n[ij]} = \frac{\exp(\alpha+(N-1-2n)\phi/2)\sinh(\phi/2)}{\sinh(\alpha+(N-1-2n)\phi/2)}a^{(N-1)}_{n[i-\binom{N-1}{n+1}j]}.$$

- *Matrix* $C^{(N)}_n$

For $\quad i \in \left\{\binom{N-1}{n-1}+1,\ldots,\binom{N}{n-1}\right\},\quad j \in \left\{\binom{N-1}{n}+1,\ldots,\binom{N}{n}\right\}$:

$$c^{(N)}_{n[ij]} = \mathrm{e}^{\phi/2}c^{(N-1)}_{n-1[i-\binom{N-1}{n-1}j]}$$

For $\quad i \in \left\{\binom{N-1}{n-1}+1,\ldots,\binom{N}{n-1}\right\},\quad j \in \left\{1,\ldots,\binom{N-1}{n}\right\}$:

$$c^{(N)}_{n[ij]} = 0,$$

For $\quad i \in \left\{1,\ldots,\binom{N-1}{n-1}\right\},\quad j \in \left\{1,\ldots,\binom{N-1}{n}\right\}$:

$$c^{(N)}_{n[ij]} = \exp(-\phi/2)\frac{\sinh(\alpha+(N-1-2n)\phi/2)}{\sinh(\alpha+(N+1-2n)\phi/2)}c^{(N-1)}_{n[ij]},$$

For $\quad i \in \left\{1,\ldots,\binom{N-1}{n-1}\right\}$:

$$c^{(N)}_{n[ii+\binom{N-1}{n}]} = -\frac{\exp(-\alpha-(N+1-2n)\phi/2)\sinh(\phi/2)}{\sinh(\alpha+(N+1-2n)\phi/2)} \times \left(a^{(N-1)}_{n-1[ii]}+\cosh(\alpha+(N+1-2n)\phi/2)\right),$$

For $\quad i \in \{1,\ldots,\binom{N-1}{n-1}\}, j \in \left\{\binom{N-1}{n}+1,\ldots,\binom{N}{n}\right\},\quad i \neq j-\binom{N-1}{n}$:

$$c^{(N)}_{n[ij]} = -\frac{\exp(-\alpha-(N+1-2n)\phi/2)\sinh(\phi/2)}{\sinh(\alpha+(N+1-2n)\phi/2)}a^{(N-1)}_{n-1[ij-\binom{N-1}{n}]}.$$

Appendix B. Zeros and poles of the rational functions for $N = 2$

$$u_{1[1]} = -\cosh\alpha,$$

$$v = \frac{\sinh(\alpha+2\alpha^*+\phi/2)+\sinh(\alpha-2\alpha^*+\phi/2)+\sinh(2\alpha+\alpha^*+3\phi/2)+\sinh(2\alpha-\alpha^*+3\phi/2)}{2(\sinh(\alpha-\alpha^*-\phi/2)+\sinh(\alpha+\alpha^*-\phi/2)+\sinh(2\alpha+\phi/2)-3\sinh(\phi/2))}$$
$$+\frac{\sinh(\alpha+\phi/2)-3\sinh(\alpha-\phi/2)+\sinh(\alpha-3\phi/2)-3\sinh(\alpha+3\phi/2)+3\sinh(\alpha^*+\phi/2)-3\sinh(\alpha^*-\phi/2)}{2(\sinh(\alpha-\alpha^*-\phi/2)+\sinh(\alpha+\alpha^*-\phi/2)+\sinh(2\alpha+\phi/2)-3\sinh(\phi/2))},$$

$$u^{(\pm)}_{1[2]} = \frac{U\pm 2\sinh(\phi/2)\sqrt{V}}{4(\cosh(\phi/2)-\cosh(2\alpha+\phi/2))},$$

$$u_{2[1]} = \frac{\sinh(\alpha-\alpha^*-\phi/2)+\sinh(\alpha+\alpha^*-\phi/2)-\sinh(\phi/2)-\sinh(3\phi/2)}{2\sinh(\alpha+\phi/2)},$$

where

$$U = 4\cosh(\alpha+\phi/2)+2\cosh(\alpha^*+\phi/2)+2\cosh(\alpha^*-\phi/2)-2\cosh(\alpha+3\phi/2)$$
$$-2\cosh(\alpha-\phi/2)-\cosh(\alpha^*+2\alpha-\phi/2)-\cosh(\alpha^*-2\alpha+\phi/2)$$

$$
\begin{aligned}
&\quad - \cosh(\alpha^* + 2\alpha + 3\phi/2) - \cosh(\alpha^* - 2\alpha - 3\phi/2),\\
V &= (\cosh(2\alpha) + \cosh(2\alpha + \phi) - 2)(8 + 4\cosh(\alpha + \alpha^*) + 4\cosh(\alpha - \alpha^*)\\
&\quad + 4\cosh(\alpha + \alpha^* + \phi) + 4\cosh(\alpha - \alpha^* + \phi) + 4\cosh(\phi) - 2\cosh(2\alpha)\\
&\quad + 2\cosh(2\alpha + \phi) + \cosh(2\alpha + 2\alpha^*) + \cosh(2\alpha - 2\alpha^*)\\
&\quad + \cosh(2\alpha + 2\alpha^* + \phi) + \cosh(2\alpha - 2\alpha^* + \phi)).
\end{aligned}
$$

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